AN EXISTENCE ANALYSIS FOR NONLINEAR EQUATIONS IN HILBERT SPACE(1)

BY
JOHN LOCKER

1. **Introduction.** In this paper we present an existence analysis for the equation $Lx = Nx$, where L is an unbounded linear operator in a Hilbert space S, and N is a nonlinear operator in S. The conditions imposed on L are satisfied by ordinary differential operators, and consequently, the theory is applicable to existence problems of nonlinear ordinary differential equations.

The theory reduces the existence problem to the study of a finite system of equations in finitely many unknowns. This technique has had widespread application to nonlinear analysis, beginning with Poincaré [11] in his work on celestial mechanics. It has been used by Schmidt [12] and Ljapunov [9] in their work on nonlinear integral equations, and by Bartle [2], Cronin [6], and Nirenberg [10], who extended their results. The latter study an equation similar to ours, assuming L to be a bounded everywhere-defined operator in a Banach space.

Cesari [3] has recently developed an existence theory for the equation $Lx = Nx$ in a Hilbert space S. He presents a system of axioms for the existence of linear operators H and P. Using these operators, he also reduces the existence problem to solving a finite system of equations in finitely many unknowns.

The theory of this paper is closely related to Cesari's theory, coinciding with his if L is a selfadjoint ordinary differential operator. Let S be a real Hilbert space with inner product (x, y) and norm $\|x\|$. The symbols $\mathscr{D}(L)$ and $\mathscr{R}(L)$ denote the domain and range, respectively, of any operator L defined in S. If L is linear, $\mathscr{N}(L)$ denotes the null space of L and L^* denotes the adjoint of L in case $\mathscr{D}(L)$ is dense in S.

Let L be a closed linear operator in S with the following properties:

(Ia) $\mathscr{D}(L)$ is dense in S,

(Ib) $\mathscr{R}(L)$ is closed in S,

(Ic) $\dim \mathscr{N}(L) = p < \infty$ and $\dim \mathscr{N}(L^*) = q < \infty$.

Let N be an operator in S with $\mathscr{D}(L) \cap \mathscr{D}(N) \neq \varnothing$, and consider the equation

$$Lx = Nx. \tag{1}$$

Under the assumptions on L we show that there exist linear operators H, P, and Q with properties analogous to the properties of Cesari's operators H and P.

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Utilizing these operators, we establish the existence of at least one solution $\hat{x} \in S$ to equation (1) provided a set of inequalities (which relate L and N) can be satisfied and provided a finite system of equations in finitely many unknowns is solvable (see Theorem 3), and also obtain estimates on the norm of such a solution $\hat{x}$. For the case when the system of equations has either more or the same number of unknowns as equations, we present two existence theorems (see Theorem 4 and Theorem 5).

To conclude this paper we illustrate our theory by studying the nonlinear boundary value problem:

$$x''(t)+x(t)+\alpha x^2(t) = \beta t, \qquad 0 \leqq t \leqq 2\pi,$$
$$x(0) = 0,$$

where α and β are constants. We show that this equation has a solution if $|\alpha| \leqq 1$, $|\beta| \leqq .001$, and obtain estimates on the norm of such a solution.

In another paper we shall examine our existence theory when L is an ordinary differential operator on a finite interval $[a, b]$ and S is the Hilbert space $L_2[a, b]$. For this case the theory assumes a specialized form which is convenient for practical applications to nonlinear ordinary differential equations.

2. **The existence analysis.** Choose elements $\phi_1, \ldots, \phi_p$ in $\mathscr{D}(L)$ to form an orthonormal base for $\mathscr{N}(L)$; choose elements $\omega_1, \ldots, \omega_q$ in $\mathscr{D}(L^*)$ to form an orthonormal base for $\mathscr{N}(L^*)$. Letting $\mathscr{N}(L)^\perp$ denote the orthogonal complement of $\mathscr{N}(L)$ in S, we note that the restriction of L to $\mathscr{D}(L) \cap \mathscr{N}(L)^\perp$ is a 1-1 closed operator having the same range as L. Let H denote the inverse of this operator:

$$H = [L|\mathscr{D}(L) \cap \mathscr{N}(L)^\perp]^{-1}. \tag{2}$$

By the Closed Graph Theorem H is a 1-1 continuous linear operator, and clearly $\mathscr{D}(H)=\mathscr{R}(L)$, $\mathscr{R}(H)=\mathscr{D}(L) \cap \mathscr{N}(L)^\perp$, and

$$LHy = y \quad \text{for all } y \in \mathscr{R}(L), \tag{3}$$

$$HLx = x-\sum_{i=1}^{p} (x, \phi_i)\phi_i \quad \text{for all } x \in \mathscr{D}(L). \tag{4}$$

Thus, H is a continuous right inverse for L. It plays a major role in our existence theory.

Let m be an integer with $m \geqq q$, and choose elements $\omega_{q+1}, \ldots, \omega_m$ in $\mathscr{D}(L^*)$ such that the elements $\omega_1, \ldots, \omega_m$ form an orthonormal set in S. Since S is the orthogonal direct sum of $\mathscr{N}(L^*)$ and $\mathscr{R}(L)$, the elements $\omega_{q+1}, \ldots, \omega_m$ belong to $\mathscr{R}(L)$, and hence, the elements $H\omega_{q+1}, \ldots, H\omega_m \in \mathscr{D}(L) \cap \mathscr{N}(L)^\perp$ can be formed.

Let S_0 be the subspace spanned by the elements $\phi_1, \ldots, \phi_p$, and $H\omega_{q+1}, \ldots, H\omega_m$. Since these elements are linearly independent, S_0 has dimension $p+m-q$. Henceforth, we assume that S_0 is a subset of $\mathscr{D}(N)$, which implies that S_0 is a subset of $\mathscr{D}(L) \cap \mathscr{D}(N)$.

Let P and Q be the projection operators defined in S by

$$Px = \sum_{i=1}^{m} (x, \omega_i)\omega_i \quad \text{for all } x \in S, \tag{5}$$

and

$$Qx = \sum_{i=1}^{p} (x, \phi_i)\phi_i + \sum_{i=q+1}^{m} (x, L^*\omega_i)H\omega_i \quad \text{for all } x \in S. \tag{6}$$

These two operators have the following properties: (a) they are continuous linear operators defined on all of S, (b) $\mathscr{R}(P)$ is the subspace $\langle \omega_1, \ldots, \omega_m \rangle$ spanned by $\omega_1, \ldots, \omega_m$, and $\mathscr{R}(P) \subseteq \mathscr{D}(L^*)$, (c) $\mathscr{R}(Q) = S_0 \subseteq \mathscr{D}(L)$, and (d) $P^2 = P$, $Q^2 = Q$. Also, the range of $I-P$ is a subset of $\mathscr{R}(L)$, and hence, $H(I-P)$ is a continuous linear operator defined on all of S.

THEOREM 1. *The following properties are valid*:

(a) $H(I-P)Lx = (I-Q)x$ *for all* $x \in \mathscr{D}(L)$.

(b) $LH(I-P)x = (I-P)x$ *for all* $x \in S$.

(c) $LQx = PLx$ *for all* $x \in \mathscr{D}(L)$.

(d) $QH(I-P)x = 0$ *for all* $x \in S$.

Proof. To show (a), take $x \in \mathscr{D}(L)$. Then

$$\begin{aligned}(I-P)Lx &= Lx - \sum_{i=1}^{m} (Lx, \omega_i)\omega_i \\ &= Lx - \sum_{i=q+1}^{m} (x, L^*\omega_i)\omega_i,\end{aligned}$$

and hence, by (4) we have

$$\begin{aligned}H(I-P)Lx &= HLx - \sum_{i=q+1}^{m} (x, L^*\omega_i)H\omega_i \\ &= (I-Q)x.\end{aligned}$$

(b) follows from (3), while (c) and (d) can be shown by direct computation.

The properties listed in Theorem 1 are analogous to the properties satisfied by Cesari's operators in [3]. We use these properties to develop an existence theory for equation (1).

Suppose $x \in \mathscr{D}(L) \cap \mathscr{D}(N)$ with $Lx = Nx$. By Theorem 1(a) we have

$$x = x^* + H(I-P)Nx \tag{7}$$

where $x^* = Qx \in S_0$. Let us try to reverse this argument. Take $x^* \in S_0$ and suppose there exists $x \in \mathscr{D}(N)$ satisfying (7). Clearly $x \in \mathscr{D}(L)$, and by Theorem 1(d) we have $Qx = Qx^* = x^*$. Thus, $Lx = LQx + LH(I-P)Nx$. Using parts (b) and (c) of Theorem 1, we get

$$Lx - Nx = P(Lx - Nx). \tag{8}$$

Therefore, x is a solution of (1) if and only if

$$P(Lx - Nx) = 0. \tag{9}$$

We have shown that if $x \in \mathscr{D}(N)$ is a solution of equation (7) corresponding to $x^* \in S_0$ and if x is a solution of equation (9), then x is a solution of the original equation (1). Equation (7) is called the *auxiliary equation.*

In the next section we introduce sufficient conditions for the existence of a unique solution x to the auxiliary equation (7) corresponding to each x^* belonging to a subset V of S_0. Then in the following section we establish sufficient conditions that there exist $x^* \in V$ such that the corresponding element x also satisfies equation (9), and hence, yields a solution to equation (1).

3. **The auxiliary equation.** Let S' be a subspace in S and let μ be a seminorm defined in S'. We assume that the following condition is satisfied:

(IIa) $\mathscr{D}(L)$ is a subset of S'.

In applications to ordinary differential equations S is the Hilbert space $L_2[a, b]$, L is a differential operator in S whose domain $\mathscr{D}(L)$ consists of functions which are at least continuous, S' is the set of functions in S which are bounded almost everywhere, and μ is the uniform norm on S'. For this case condition (IIa) is certainly satisfied.

We assume that the following condition is satisfied:

(IIb) There exist constants $k \geqq 0$ and $k' \geqq 0$ such that

$$\|H(I-P)x\| \leqq k\|x\|, \qquad \mu(H(I-P)x) \leqq k'\|x\| \quad \text{for all } x \in S.$$

Choose an element $x_0 \in S_0$. Noting that $x_0 \in \mathscr{D}(L) \cap \mathscr{D}(N)$, let $y = H(I-P)Nx_0$. Choose constants e and e' such that $\|y\| \leqq e$, $\mu(y) \leqq e'$. Let c, d, r, and R_0 be real numbers with $0 < c < d$ and $0 < r < R_0$, and define sets V and $\tilde{S}_0$ in S by

$$V = \{x \in S_0 \mid \|x - x_0\| \leqq c,\ \mu(x-x_0) \leqq r\} \tag{10}$$

and

$$\tilde{S}_0 = \{x \in S' \mid \|x - x_0\| \leqq d,\ \mu(x-x_0) \leqq R_0\}. \tag{11}$$

Clearly $x_0 \in V \subseteq \tilde{S}_0$, so these sets are nonempty. For each $x^* \in V$ let

$$S(x^*) = \{x \in S' \mid Qx = x^*,\ \|x-x_0\| \leqq d,\ \mu(x-x_0) \leqq R_0\}. \tag{12}$$

Clearly $x^* \in S(x^*)$, so each of the sets $S(x^*)$ is nonempty, and $x^* \in S(x^*) \subseteq \tilde{S}_0$ for all $x^* \in V$.

Finally, we assume the following two conditions are satisfied:

(IIc) The set $\tilde{S}_0$ is a subset of $\mathscr{D}(N)$, and there exists a constant $l \geqq 0$ such that

$$\|Nx - Ny\| \leqq l\|x-y\| \quad \text{for all } x, y \in \tilde{S}_0.$$

(IId) For each $x^* \in V$ the set $S(x^*)$ is closed in S.

THEOREM 2. *If conditions* (Iabc) *and* (IIabcd) *are satisfied and if*

$$kl < 1, \quad c+e \leqq (1-kl)d, \quad r+e' \leqq R_0 - k'ld, \tag{13}$$

then for each $x^* \in V$ there exists a unique element $x \in S(x^*)$ which is a solution to the auxiliary equation (7) *corresponding to x^*. Furthermore, $x \in \mathscr{D}(L) \cap \mathscr{D}(N)$, $Qx = x^*$, and $Lx - Nx = P(Lx - Nx)$. Also, the solutions x vary continuously with the x^*.*

Proof. Fix $x^* \in V$, and let $T: S(x^*) \to S'$ be the operator defined by

$$Tx = x^* + H(I-P)Nx \quad \text{for all } x \in S(x^*).$$

If $x \in S(x^*)$ and $y = Tx$, then $y \in S'$ with $Qy = x^*$ by Theorem 1(d),

$$\begin{aligned} \|y - x_0\| &= \|x^* - x_0 + H(I-P)Nx - H(I-P)Nx_0 + \gamma\| \\ &\leqq c + k\|Nx - Nx_0\| + e \\ &\leqq d, \end{aligned}$$

and similarly, $\mu(y - x_0) \leqq R_0$. Thus, T maps $S(x^*)$ into itself. Note that T is a contraction, and hence, by the Banach Fixed Point Theorem the auxiliary equation (7) is uniquely solvable in $S(x^*)$. Also, if $x^* \in V$ and $y^* \in V$, and if $x \in S(x^*)$ and $y \in S(y^*)$ are the unique elements with

$$x = x^* + H(I-P)Nx, \qquad y = y^* + H(I-P)Ny,$$

then

$$\begin{aligned} \|x - y\| &\leqq \|x^* - y^*\| + \|H(I-P)(Nx - Ny)\| \\ &\leqq \|x^* - y^*\| + kl\|x - y\|, \end{aligned}$$

or $\|x-y\| \leqq (1-kl)^{-1}\|x^* - y^*\|$. This completes the proof of the theorem.

Theorem 2 guarantees that the auxiliary equation (7) can be solved for each $x^* \in V$. In fact, it permits us to set up a correspondence between $x^* \in V$ and the solution $x \in S(x^*)$ of the auxiliary equation: under the hypothesis of Theorem 2 let $\Gamma: V \to \mathscr{D}(L) \cap \tilde{S}_0$ be the continuous operator defined by $\Gamma(x^*) = x$ for $x^* \in V$ where x is the unique element in $S(x^*)$ which is a solution to the auxiliary equation (7) corresponding to x^*.

Note that $\Gamma(x^*) \in \mathscr{D}(L) \cap \mathscr{D}(N)$ for each $x^* \in V$, and hence, $P(L\Gamma x^* - N\Gamma x^*)$ is an operator mapping V into the subspace $\langle \omega_1, \ldots, \omega_m \rangle$. The next theorem is an immediate consequence of Theorem 2.

THEOREM 3. *Let conditions* (Iabc) *and* (IIabcd) *be satisfied and let relations* (13) *be valid. If there exists an element* $x^* \in V$ *such that*

$$P(L\Gamma x^* - N\Gamma x^*) = 0, \tag{14}$$

then the element $\hat{x} = \Gamma x^*$ *is a solution of the equation* $Lx = Nx$, $Q\hat{x} = x^*$, *and* $\|\hat{x} - x_0\| \leqq d$, $\mu(\hat{x} - x_0) \leqq R_0$.

In Theorem 3 the problem of solving equation (1) has been reduced to the problem of solving equation (14), which is actually a system of m equations in $p+m-q$ unknowns. Equation (14) is called the *bifurcation equation* or the *determining equation.* We examine it in the next section.

4. **The bifurcation equation.** In this section sufficient conditions are introduced for the existence of a solution $x^* \in V$ to the bifurcation equation (14). Let $\psi: \mathscr{D}(L) \cap \mathscr{D}(N) \to \langle \omega_1, \ldots, \omega_m \rangle$ be the operator defined by

$$\psi x = P(Lx - Nx) \quad \text{for all } x \in \mathscr{D}(L) \cap \mathscr{D}(N). \tag{15}$$

Note that if conditions (IIac) are satisfied, then V and $\mathscr{D}(L) \cap \tilde{S}_0$ are both subsets of $\mathscr{D}(L) \cap \mathscr{D}(N)$, and for elements x, y belonging to $\mathscr{D}(L) \cap \tilde{S}_0$:

$$\begin{aligned}\|\psi x-\psi y\| &= \left\|\sum_{i=1}^{m} (Lx-Ly, \omega_i)\omega_i - \sum_{i=1}^{m} (Nx-Ny, \omega_i)\omega_i\right\| \\ &\leqq \sum_{i=1}^{m} |(x-y, L^*\omega_i)| + \sum_{i=1}^{m} |(Nx-Ny, \omega_i)| \\ &\leqq \left[\sum_{i=q+1}^{m} \|L^*\omega_i\| + ml\right] \|x-y\|.\end{aligned}$$

Throughout the remainder of this section we assume that conditions (Iabc) and (IIabcd) are satisfied and that relations (13) are valid. Thus, the continuous operators Γ, $\psi|\mathscr{D}(L) \cap \tilde{S}_0$, and $\psi|V$ exist with

$$V \xrightarrow{\Gamma} \mathscr{D}(L) \cap \tilde{S}_0 \xrightarrow{\psi} \langle \omega_1, \ldots, \omega_m \rangle$$

and

$$V \xrightarrow{\psi} \langle \omega_1, \ldots, \omega_m \rangle.$$

Note that $\psi\Gamma$ maps the "ball" V, which is a subset of the $p+m-q$ dimensional space S_0, continuously into the m dimensional space $\langle \omega_1, \ldots, \omega_m \rangle$, and also that the bifurcation equation (14) can be rewritten as

$$(14)' \qquad \psi\Gamma x^* = 0.$$

The operator $\psi\Gamma$ is difficult to work with because Γ is defined by an iteration process. On the other hand, $\psi|V$ is easily obtained. The following lemma relates these two operators.

LEMMA. *Let conditions* (Iabc) *and* (IIabcd) *be satisfied, and let relations* (13) *be valid. Then* $\|\psi\Gamma x^* - \psi x^*\| \leqq (kld+e)l$ *for all* $x^* \in V$.

Proof. Take $x^* \in V$ and let $x = \Gamma x^*$. Then $x \in S(x^*)$, $Qx = Qx^*$, and $PLx = PLx^*$, so $\psi\Gamma x^* - \psi x^* = P(Nx^* - Nx)$. Hence, by Bessel's inequality and (IIc) we have

$$\begin{aligned}\|\psi\Gamma x^* - \psi x^*\| &\leqq l\|x - x^*\| \\ &\leqq l\|H(I-P)Nx - H(I-P)Nx_0 + \gamma\| \\ &\leqq (kld+e)l.\end{aligned}$$

We use this lemma to determine conditions on $\psi|V$ which guarantee that the bifurcation equation (14) is solvable.

Apply the Gram-Schmidt process to the elements $H\omega_{q+1}, \ldots, H\omega_m$ to obtain orthonormal elements $\eta_{q+1}, \ldots, \eta_m$. Let $M = p+m-q$, and let E^M be a copy of Euclidean M-space where we represent each point $\xi \in E^m$ as an M-tuple: $\xi = (b_1, \ldots, b_p, c_{q+1}, \ldots, c_m)$. Also, let E^m be a copy of Euclidean m-space where we represent each point $u \in E^m$ as an m-tuple: $u=(u_1, \ldots, u_m)$. We define two operators $\Gamma_1: E^M \to S_0$ and $\Gamma_2: \langle \omega_1, \ldots, \omega_m \rangle \to E^m$ by

$$\Gamma_1(b_1, \ldots, b_p, c_{q+1}, \ldots, c_m) = \sum_{i=1}^{p} b_i\phi_i + \sum_{i=q+1}^{m} c_i\eta_i \tag{16}$$

and

$$\Gamma_2\left(\sum_{i=1}^{m} u_i\omega_i\right) = (u_1, \ldots, u_m). \tag{17}$$

Clearly Γ_1 and Γ_2 are isomorphisms. Let $\xi_0 \in E^M$ be the element with $\Gamma_1(\xi_0)=x_0$, and let $\Psi: E^M \to E^m$ be the operator

$$\Psi = \Gamma_2\psi\Gamma_1. \tag{18}$$

Choose a number $\varepsilon > 0$ such that the set

$$U = \{\xi \in E^M \mid \|\xi - \xi_0\| \leqq \varepsilon\} \tag{19}$$

is mapped by Γ_1 into the set V. The existence of such an ε is not difficult to show. We observe that the operators $\Gamma_2\psi\Gamma_1$ and $\Gamma_2\psi\Gamma\Gamma_1$ map the ball U, which is a subset of E^M, continuously into E^m. This is used in the next theorem to obtain an existence theorem for equation (1) for the case that E^m has dimension 1.

THEOREM 4. *Let $m=1$, let conditions* (Iabc) *and* (IIabcd) *be satisfied, and let relations* (13) *be valid. If there exists a number $\delta > 0$ such that the interval $[-\delta, \delta]$ is a subset of $\Psi(U)$ and if $(kld+e)l \leqq \delta$, then there exists $x^* \in V$ such that the element $\hat{x}=\Gamma(x^*)$ is a solution of the equation $Lx=Nx$, and $Q\hat{x}=x^*$, $\|\hat{x}-x_0\| \leqq d$, $\mu(\hat{x}-x_0) \leqq R_0$.*

Proof. Choose $\xi_1, \xi_2 \in U$ such that $\Psi(\xi_1)=\delta$ and $\Psi(\xi_2)=-\delta$. Let $x_1^*=\Gamma_1(\xi_1)$, $x_2^*=\Gamma_1(\xi_2)$. Clearly x_1^* and x_2^* are elements of V, and by the lemma we have $\|\psi\Gamma x_i^* - \psi x_i^*\| \leqq \delta$ for $i=1, 2$. Thus,

$$\begin{aligned} |\Gamma_2\psi\Gamma\Gamma_1(\xi_1) - \delta| &= |\Gamma_2\psi\Gamma\Gamma_1(\xi_1) - \Gamma_2\psi\Gamma_1(\xi_1)| \\ &= \|\psi\Gamma x_1^* - \psi x_1^*\| \leqq \delta, \end{aligned}$$

or $\Gamma_2\psi\Gamma\Gamma_1(\xi_1) \geqq 0$. Similarly, $\Gamma_2\psi\Gamma\Gamma_1(\xi_2) \leqq 0$. Since $\Gamma_2\psi\Gamma\Gamma_1(U)$ is connected, there exists $\xi \in U$ such that $\Gamma_2\psi\Gamma\Gamma_1(\xi)=0$. If we set $x^*=\Gamma_1(\xi)$, then $x^* \in V$ and $P(L\Gamma x^* - N\Gamma x^*)=0$. The proof is completed using Theorem 3.

To conclude this section we give another existence theorem which relaxes the condition $m=1$. Let conditions (Iabc) and (IIabcd) be satisfied, and let relations (13) be valid. In addition we assume that the following conditions are satisfied:

(IIIa) $p \geqq q$,

(IIIb) $\Psi(\xi_0)=0$,

(IIIc) The first order partial derivatives of Ψ exist and are continuous on U,

(IIId) The Jacobian matrix for Ψ has rank m at ξ_0.

The first condition is equivalent to the condition $M \geqq m$, implying that Ψ maps from a high dimensional space into a lower dimensional space. The second condition says that $P(Lx_0 - Nx_0)=0$, which means that x_0 can be considered as an approximate solution to equation (1). In applications this suggests how one should choose x_0. The third condition corresponds to putting certain differentiability conditions on the operator N. The last condition guarantees that the range of Ψ covers a neighborhood of the origin in E^m. In fact, by the Inverse Function Theorem we can choose a number $\delta>0$ such that the set

$$W = \{u \in E^m \mid \|u\| \leqq \delta\} \tag{20}$$

is a subset of $\Psi(U)$, and such that there exists a continuous mapping $\Lambda\colon W \to U$ with $\Psi[\Lambda(u)]=u$ for all $u \in W$.

THEOREM 5. *Let conditions* (Iabc), (IIabcd), *and* (IIIabcd) *be satisfied, and let relations* (13) *be valid. If* $\delta>0$ *is a number chosen as above and if* $(kld+e)l<\delta$, *then there exists* $x^* \in V$ *such that the element* $\hat{x}=\Gamma(x^*)$ *is a solution of the equation* $Lx=Nx$, *and* $Q\hat{x}=x^*$, $\|\hat{x}-x_0\|\leqq d$, $\mu(\hat{x}-x_0)\leqq R_0$.

Proof. Consider the two continuous maps $\Gamma_2\psi\Gamma\Gamma_1\Lambda\colon W\to E^m$ and $I\colon W\to E^m$ where $I(u)\equiv u$. Take $u\in W$ and let $x^*=\Gamma_1\Lambda(u)$. Then $x^*\in V$, $\Gamma_2\psi(x^*)=I(u)$, and

$$\|\Gamma_2\psi\Gamma\Gamma_1\Lambda(u)-I(u)\| = \|\psi\Gamma(x^*)-\psi(x^*)\| \leqq (kld+e)l,$$

or

$$\|\Gamma_2\psi\Gamma\Gamma_1\Lambda(u)-I(u)\| < \delta \quad \text{for all } u\in W.$$

This inequality implies that for each $u\in\partial W$, the line segment joining $I(u)$ and $\Gamma_2\psi\Gamma\Gamma_1\Lambda(u)$ does not contain the origin of E^m. By the Poincaré-Bohl Theorem [6, p. 32] the local degree of $\Gamma_2\psi\Gamma\Gamma_1\Lambda$ at 0 relative to W is equal to the local degree of I at 0 relative to W: $d(\Gamma_2\psi\Gamma\Gamma_1\Lambda, W, 0)=d(I, W, 0)$. But $d(I, W, 0)=1$, and hence, $d(\Gamma_2\psi\Gamma\Gamma_1\Lambda, W, 0)\neq 0$. Therefore, there exists $u\in W$ such that $\Gamma_2\psi\Gamma\Gamma_1\Lambda(u)=0$. Setting $x^*=\Gamma_1\Lambda(u)$, we have $x^*\in V$ and $P(L\Gamma x^*-N\Gamma x^*)=0$. The proof is completed using Theorem 3.

5. **An application.** We illustrate our existence theory by studying the nonlinear boundary value problem:

$$\begin{aligned} x''(t)+x(t)+\alpha x^2(t) &= \beta t, \quad 0\leqq t\leqq 2\pi,\\ x(0) &= 0, \end{aligned} \tag{21}$$

where α and β are real constants. Let I be the interval $[0, 2\pi]$, let S be the real Hilbert space $L_2(I)$, let S' be the subspace in S consisting of all functions which are bounded a.e., and let μ be the uniform norm in S', i.e., $\mu(x)=\inf\{c \mid |x|\leqq c \text{ a.e.}\}$.

We denote by $H^2(I)$ the subspace of S consisting of all functions $x(t)$ with the properties: x is continuous on I, x' exists and is absolutely continuous on I, $x''\in S$. Let L be the 2nd order differential operator in S defined by $Lx=x''+x$ where the domain $\mathscr{D}(L)$ consists of all functions $x\in H^2(I)$ with $x(0)=0$. It is well known [13, pp. 431–434] that L is a closed linear operator in S with dense domain and closed range, and that the adjoint L^* is the 2nd order differential operator in S given by $L^*x=x''+x$ where $\mathscr{D}(L^*)$ consists of all functions $x\in H^2(I)$ with $x(0)=x(2\pi)=0$, $x'(2\pi)=0$. Since $\mathscr{N}(L)$ and $\mathscr{N}(L^*)$ are subsets of $\langle \sin t, \cos t\rangle$, both null spaces are finite-dimensional. In fact, it is easy to check that $\mathcal{N}(L)=\langle \sin t\rangle$ and $\mathcal{N}(L^*)=\langle 0\rangle$. Thus, L is a closed linear operator in S with properties (Iabc), $p=1$, $q=0$, and $\mathscr{R}(L)=S$.

Let N be the operator in S defined by $\mathscr{D}(N)=S'$, $Nx=-\alpha x^2(t)+\beta t$ for all $x(t)\in\mathscr{D}(N)$. Note that $\mathscr{D}(L)\cap\mathscr{D}(N)=\mathscr{D}(L)$, which is a subset of S'. The equation $Lx=Nx$ is equivalent to the nonlinear boundary value problem (21). Using Theorem 4, we show that there exists a solution $\hat{x}$ to the equation $Lx=Nx$ for all (α,β) with $|\alpha|\leqq 1$, $|\beta|\leqq .001$.

Choose $\phi_1(t)=\pi^{-1/2}\sin t$, and let $G(t,s)=\sin t\cos s-\cos t\sin s$. Then the right inverse operator H can be shown to have the integral representation

$$Hy(t) = \int_0^t G(t,s)y(s)\,ds+\sin t\cdot\int_0^{2\pi} g(s)y(s)\,ds,\quad 0\leqq t\leqq 2\pi, \tag{22}$$

for all $y\in S$, where $g(t)=-\cos t+(t/2\pi)\cos t-(1/2\pi)\sin t$.

Next, observe that the function $\omega(t)=(t-2\pi)\sin t$ belongs to $\mathscr{D}(L^*)$: choose $m=1$ and $\omega_1(t)=\|\omega\|^{-1}\omega(t)$. For this choice of m and ω_1, $M=p+m-q=2$ and the operators P, Q, ψ, Γ_1, Γ_2, and Ψ are given by (5), (6), (15), (16), (17), and (18), respectively. In particular, $\Psi\colon E^2\to E^1$ is given by

$$\Psi(b_1,c_1) = .502738c_1-.211427\alpha b_1^2+.093851\alpha b_1c_1+.033884\alpha c_1^2. \tag{23}$$

Also, the subspace $S_0=\langle\phi_1, H\omega_1\rangle$ has dimension 2 and is a subset of $\mathscr{D}(N)$.

The element $x_0\in S_0$ is chosen so that $\psi(x_0)=0$ or $\Psi(\xi_0)=0$. From (23) we note that the latter condition is satisfied for $\xi_0=(0,0)$, and this choice of ξ_0 yields $x_0(t)\equiv 0$. Then $\gamma=H(I-P)Nx_0$ is given by $\gamma(t)=2\beta\sin t+\beta t$ with $\|\gamma\|=8.373592|\beta|$ and $\mu(\gamma)=2\pi|\beta|$. Let $e=8.374|\beta|$ and $e'=6.284|\beta|$.

It is clear that condition (IIa) is satisfied. By means of (22) the operator $H(I-P)$ can also be shown to be an integral operator of the form

$$H(I-P)x(t) = \int_0^{2\pi} K_1(t,s)x(s)\,ds,\quad 0\leqq t\leqq 2\pi,$$

for all $x\in S$. Thus, condition (IIb) is satisfied provided

$$k \geqq \left(\int_0^{2\pi}\int_0^{2\pi}[K_1(t,s)]^2\,ds\,dt\right)^{1/2} = 1.413573$$

and

$$k' \geqq \left(\max_{0\leqq t\leqq 2\pi}\int_0^{2\pi}[K_1(t,\xi)]^2\,d\xi\right)^{1/2} = .832194.$$

Let $k=1.414$ and $k'=.833$. For condition (IIc) the set $\tilde{S}_0$ is given by

$$\tilde{S}_0 = \{x\in S' \mid \|x\|\leqq d,\ \mu(x)\leqq R_0\},$$

which is a subset of $\mathscr{D}(N)$, and for $x\in\tilde{S}_0$, $y\in\tilde{S}_0$ we have

$$\begin{aligned}|Nx(t)-Ny(t)| &= |\alpha|\,|y(t)+x(t)|\,|y(t)-x(t)|\\ &\leqq 2R_0|\alpha|\,|x(t)-y(t)|\end{aligned}$$

and $\|Nx-Ny\|\leqq 2R_0|\alpha|\,\|x-y\|$. Let $l=2R_0|\alpha|$. Condition (IId) has been shown to hold by Cesari [3, p. 404]. Thus, conditions (IIabcd) are satisfied for these choices of k, k', and l.

For any function $x(t) \in S_0$ with $\|x\| \leqq c$, we can show that $|x(t)| \leqq 3.049797c$. Hence, setting $r = 3.049797c$ and $\varepsilon = c$, the sets V and U simplify to

$$V = \{x \in S_0 \mid \|x\| \leqq c\} \quad \text{and} \quad U = \{\xi \in E^2 \mid \|\xi\| \leqq c\}.$$

If we assume that $|\alpha| \leqq 1$ and set $\delta = .502738c - .033884c^2$, then from (23) we obtain $\Psi(0, c) \geqq \delta$ and $\Psi(0, -c) \leqq -\delta$; under these assumptions the interval $[-\delta, \delta]$ is a subset of $\Psi(U)$.

Finally, to apply Theorem 4, we need to determine a bound on the parameter β and choose the numbers c, d, and R_0 such that relations (13) are valid and such that $(kld+e)l \leqq \delta$:

$$\begin{aligned}
&0 < c < d, \qquad 3.049797c < R_0,\\
&(1.414)(2R_0) < 1,\\
(24) \qquad &c + 8.374|\beta| + (1.414)(2R_0)d \leqq d,\\
&3.049797c + 6.284|\beta| + (.833)(2R_0)d \leqq R_0,\\
&[(1.414)(2R_0)d + 8.374|\beta|](2R_0) \leqq .502738c - .033884c^2.
\end{aligned}$$

One solution of these inequalities is given by $|\alpha| \leqq 1$, $|\beta| \leqq .001$, $c = .01$, $d = .03$, and $R_0 = .1$, which yields $r = .030498$, $l = .2|\alpha|$, $\varepsilon = .01$, and $\delta = .005024$. From Theorem 4 we conclude that for each pair of real numbers (α, β) with $|\alpha| \leqq 1$, $|\beta| \leqq .001$ there exists a real-valued function $x(t)$ which is twice continuously differentiable on the interval $[0, 2\pi]$, and which is a solution of equation (21) with $\|x\| \leqq .03$ and $|x(t)| \leqq .1$.

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UNIVERSITY OF MICHIGAN,
ANN ARBOR, MICHIGAN